Bounds on existence of odd and unique expanders

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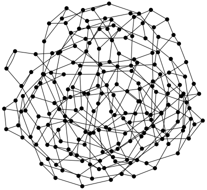
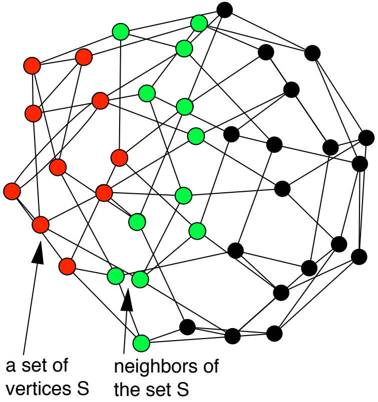


What is an expander?

Definition

We call a d-regular graph with n vertices a $\boldsymbol{(n, d, \alpha, \epsilon)}$-**expander** if every subset S of at most $\alpha \cdot n$ of its vertices has at least $\epsilon \cdot |S|$ neighbors.



Example: $(40, 4, 0.5, 0.3)$-expander

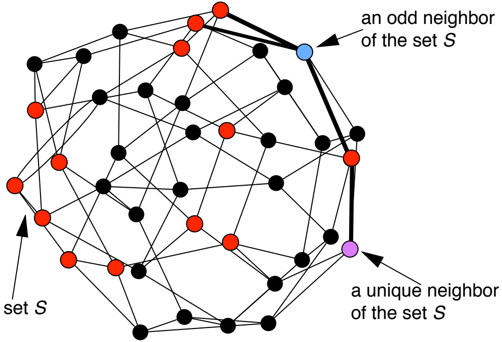
Odd and unique neighbors

Definition

Odd neighbor is attached by an *odd* number of edges to S.

Definition

Unique neighbor is attached by a *single* edge to the set S.



unique neighbor $\rightarrow$ odd neighbor $\rightarrow$ neighbor

Trade-off between α and ϵ

"... if every subset S of at most $\alpha \cdot n$ of its vertices has at least $\epsilon \cdot |S|$ neighbors."

- The expansion property ϵ is more strictly restricted if the argument α is increased. What is the general trade-off between these values?
- Is it possible to construct expander graphs with positive expansion ϵ for every $0 < \alpha < 1$, or is there any fundamental restriction?

Main results

Unique neighbor expanders

Theorem 10 (A small subset of vertices without unique neighbors)

Let $G = (V, E)$ be a graph. There exists a nonempty subset of its vertices $S \subseteq V$ such that $|S| \leq \frac{|V|}{2} + 1$, which does not have any unique neighbors.

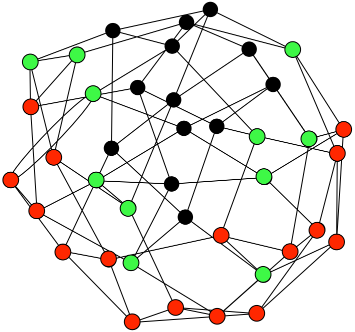


Figure: An example of a small set of vertices (red colored), to which all of its neighbors (green colored) are connected at least twice.

Theorem 10 (A small subset of vertices without unique neighbors)

Let $G = (V, E)$ be a graph. There exists a nonempty subset of its vertices $S \subseteq V$ such that $|S| \leq \frac{|V|}{2} + 1$, which does not have any unique neighbors.

- $\Rightarrow$ no infinite family of unique-expanders for $\alpha \geq 1/2$ and $\epsilon > 0$.
- For a d-regular simple graph the bound can be improved to $O\left(\frac{\log d}{d}\right)$.

Conjecture 1 (A small even subset)

Let $G = (V, E)$ be a graph. There exists a nonempty subset of its vertices $S \subseteq V$ such that $|S| \leq \frac{|V|}{2} + 1$, which does not have any *odd* neighbors.

Partial results:

- Proved for bipartite graphs.
- Problem reduced to biconnected graphs.
- Experimentally verified for small graphs (up to 12 vertices).

Further research and open problems

1. What is the general trade-off between values d, α and ϵ?
2. Is α for simple d-regular unique-expanders bounded by $O\left(1/d\right)$?
3. Are there 4 vertices without unique neighbors in every simple n-regular graph with $2n$ vertices?
4. Conjecture 1 is open for biconnected graphs, which are not bipartite.

Thank you for your attention!

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Připomínka 1

... na str. 25 je dimenze prostorů jen 1 a ne 2, jak je uvedeno. (To ale nevadí.)

Komentář

Vskutku, podstatné pro tvrzení je, že soustava $n+1$ lineárních rovnic o n neznámých mající jedno řešení má ještě alespoň jedno jiné řešení.

Připomínka 2

Na str. 16 by bylo vhodné uvést, proč platí nerovnost mezi druhým a třetím řádkem – proč se jde zbavit celé části. (Úprava je nicméně korektní.)

Komentář

Problém spočívá v nerovnosti $\left(\frac{n}{\lfloor x \rfloor}\right)^{\lfloor x \rfloor} \leq \left(\frac{n}{x}\right)^{x}$ pro $n \in \mathbb{N}$ a $0 < x < n/e$. Derivace spojité funkce $\left(\frac{n}{x}\right)^{x}$ podle proměnné x je rovna

$$\left(\frac{n}{x}\right)^{x} \cdot \left(\log\left(\frac{n}{x}\right) - 1\right)$$

a nulová pro $x = \frac{n}{e}$. Pro $0 < x < \frac{n}{e}$ je derivace kladná a zkoumaná funkce tudíž rostoucí. Uznávám, že použitá nerovnost není zřejmá, a zaslouží podrobnější vysvětlení.

Připomínka 3

A pak překlepy v tabulce na str. 28, kde zápis $0 < \alpha < 0$ vzbuzuje pochybnosti.

Komentář

Správně má být $0 < \alpha < 1$, děkuji za upozornění.